

Structure of almost Abelian Lie algebras

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Abstract

An almost Abelian Lie algebra is a non-Abelian Lie algebra with a codimension 1 Abelian subalgebra. It is known that such an algebra necessarily has a codimension 1 Abelian ideal. Almost Abelian Lie algebras are perhaps the next simplest Lie algebras after Abelian Lie algebras. The present exposition studies the structure and important algebraic properties of almost Abelian Lie algebras of arbitrary dimension over any field of scalars.

1 Introduction

In our definition an almost Abelian Lie algebra is a non-Abelian Lie algebra \mathbf{L} over a field \mathbb{F} which has a codimension 1 Abelian subalgebra. It was shown in [4] (Proposition 3.1) that such an algebra necessarily contains a codimension 1 Abelian ideal. The result is announced for fields \mathbb{F} of characteristic zero and in finite dimensions, but in fact the proof works with minor adaptations in the most general case. Often almost Abelian Lie algebras are defined as those having a codimension 1 Abelian ideal, but in view of the result above we find it more convenient to start with a codimension 1 subalgebra.

There is certain controversy regarding the notion of an almost Abelian Lie algebra in the literature. Some authors (e.g., [11], [8], [13]) define an almost Abelian Lie algebra as a Lie algebra for which there exists a basis $\{e_0, e_i\}$, $i = 1, \dots, n$ such that $[e_0, e_i] = e_i$ and $[e_i, e_j] = 0$ for all i , i.e., $\text{ad}_{e_0} = \mathbf{1}$. This definition is too strict in that it misses all other possibilities for the operator ad_{e_0} not similar to identity. On the other hand, some sources prefer to include Abelian Lie algebras in the class of almost Abelian Lie algebras (e.g., [11],[17]), but we prefer not to do so because of principal structural differences between the two.

Probably the best motivation to study almost Abelian Lie algebras comes from the theory of integrable systems and PDE. Let $(C^\infty(M), \{, \})$ be a Poisson algebra over an n -dimensional manifold M , and let $H \in C^\infty(M)$ be a Hamiltonian. The Hamiltonian system evolving according to H is Liouville integrable if H is contained in an Abelian subalgebra of $(C^\infty(M), \{, \})$ (considered as a Lie algebra) generated by at least n functionally independent integrals of motion. In particular, if H is invariant under the action of a Lie algebra \mathbf{L} of Noether symmetries which has an at least $n - 1$ -dimensional Abelian subalgebra then the system is integrable in Noether integrals. It follows that every Hamiltonian system invariant under a transitively acting (i.e., $\dim \mathbf{L} = n$ and complete) almost Abelian Lie algebra of Noether symmetries is integrable in Noether integrals. For instance, every invariant Hamiltonian system on an almost Abelian Lie group (i.e., a Lie group with an almost Abelian Lie algebra) is integrable in Noether integrals. Integrability in Noether integrals may be labelled simplistic, but the class of invariant systems on almost Abelian Lie groups is sufficiently diverse to demonstrate non-trivial phenomena as non-commutativity, non-unimodularity and curvature, which can be studied in a completely explicit manner.

Another way an almost Abelian Lie algebra arises is from a linear dynamical system. Let \mathbb{F} be a field, and \mathbf{V} an \mathbb{F} -vector space. A discrete linear dynamical system is an equation of the form

$$v(n+1) = Tv(n), \quad v : \mathbb{N}_0 \mapsto \mathbf{V},$$

where $T \in \text{End}_{\mathbb{F}}(\mathbf{V})$ is a linear operator. To give our dynamical system a Hamiltonian flavour consider the extended \mathbb{F} -vector space

$$\mathbf{L} = \mathbb{F}H \oplus_{\mathbb{F}} \mathbf{V}$$

with a distinguished element $H \in \mathbf{L}$ called the *Hamiltonian*. Make \mathbf{L} into a Lie algebra by setting

$$[u, v] = 0, \quad [H, v] = \text{ad}_H v = Tv, \quad \forall v, u \in \mathbf{V}.$$

Now if $T \neq 0$ then \mathbf{L} is an almost Abelian Lie algebra that encodes the dynamics of the original system (if $T = 0$ then \mathbf{L} is Abelian and the dynamics is trivial),

$$v(n+1) = [H, v(n)].$$

If the vector space \mathbf{V} is endowed with a topology then the continuous counterpart of the system

can be considered as well,

$$\frac{v(t)}{t} = [H, v(t)].$$

Thus an almost Abelian Lie algebra is also a stationary linear dynamical system, and the study of its algebraic properties yields to the understanding of the dynamics. Again, linear dynamical systems may be considered a textbook subject as they are usually solved in terms of matrix algebra. But things are not as trivial when the operator T is not diagonalizable, or \mathbf{V} is infinite dimensional, or when \mathbb{F} is neither \mathbb{R} nor \mathbb{C} . Matrix tools such as determinants, traces and transpositions need not exist. Moreover, even advanced methods like canonical forms (e.g., Jordan or Frobenius) become non-trivial issues and need not exist. No reference can be made to the spectral theory of Hilbert space operators because \mathbf{V} is merely a vector space. And even in the simplest case of a real or complex finite dimensional diagonalizable matrix T many algebraic aspects need to be described explicitly and in a systematic manner.

The smallest (in terms of dimensions) almost Abelian Lie algebra is the only (up to isomorphism) non-Abelian Lie algebra of dimension 2, which is the Lie algebra $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$ of the group of affine transformations on the real line. This algebra can be described in a basis $\{e_0, e_1\}$ with relations $[e_0, e_1] = e_1$, i.e., $\text{ad}_{e_0} = \mathbf{1}$. In dimension 3 the almost Abelian Lie algebras are the non-Abelian solvable Bianchi algebras Bi(II)-Bi(VII) (Bi(I) is Abelian and Bi(VIII) and Bi(IX) are simple, so we will not consider them here) which were first classified for $\mathbb{F} = \mathbb{R}$ by Bianchi [2] (see [9] for a more modern and elegant approach). The classification corresponds to the similarity classes of 2×2 matrices ad_{e_0} . The most prominent of them is the nilpotent Heisenberg algebra $\mathbf{H}_{\mathbb{F}}$ which arises from the canonical commutation relations in quantum mechanics,

$$e_0 = \frac{\partial}{\partial x}, \quad e_1 = \mathbf{1}, \quad e_2 = x, \quad [e_0, e_2] = e_1, \quad [e_0, e_1] = [e_1, e_2] = 0.$$

Thorough studies of the Heisenberg algebra and the corresponding Lie group can be found, for instance, in [6] and [19]. The remaining solvable Bianchi algebras are mainly known in the context of cosmology. They arise as the algebras of geometric symmetries in homogeneous anisotropic cosmological models ([5], [14], [16], [18] and many others). More details on the relevance of Bianchi groups in cosmology can be found in [1] and references therein. Another very interesting application of 3-dimensional solvable Lie algebras is crystallography, where these algebras and their Lie groups represent the spatial symmetries of anisotropic crystals (see, for instance, [15] and references therein).

In this paper we consider almost Abelian Lie algebras \mathbf{L} of any dimension $\dim_{\mathbb{F}} \mathbf{L} = 1 + \aleph$ over any field \mathbb{F} , so that the codimension 1 Abelian ideal has dimension \aleph . Note that we identify

index sets with their cardinalities, therefore we use the symbol \aleph to denote both the cardinality of a Hamel basis in a vector space and an actual index set of that cardinality. In order to deal with this generality certain precautions have been taken in the methodology. No integers other than 0 and ± 1 are assumed to be contained in \mathbb{F} a priori in order to avoid a potential conflict with the characteristic of \mathbb{F} . No reference is made to polynomial algebraic and spectral methods such as eigenvalues or factorizations so that algebraic and topological aspects of \mathbb{F} such as algebraic extensions or topological closedness are irrelevant. And of course, no dimensional or matrix arguments such as rank-nullity or canonical forms can be used in this generality. This may have made some otherwise trivial facts unexpectedly tricky, but the matrix intuition mostly remains valid. Some structures as homomorphisms become somewhat richer in infinite dimensions.

The paper is structured as follows. First in section 2 we introduce notations by reminding some notions and facts from the elementary Lie theory which are used in the body. This makes the paper essentially self-contained with little reference to the state-of-the-art in the field, and thereby more accessible to non-specialists. Then in section 3 we define almost Abelian Lie algebras, discuss their subalgebras and ideals, and decompose them into an indecomposable core and a central extension. Further in section 4 we describe homomorphisms between and automorphisms of almost Abelian Lie algebras which then naturally leads to a classification up to isomorphism. Afterwards in section 5 we describe derivations and Lie orthogonal operators on almost Abelian Lie algebras. Finally in section 6 we study the centre of the universal enveloping algebra of an almost Abelian Lie algebra.

2 Lie algebras

Lie algebras will be the main subject of this exposition, so let us collect some definitions, notations and facts including proofs where the latter are particularly simple. For a thorough exposition of Lie algebras and Lie groups we refer to the all time classical monograph by Knapp [12].

A Lie algebra is a vector space \mathbf{L} (over a field \mathbb{F}) along with an antisymmetric bilinear map $[\cdot, \cdot] : \mathbf{L} \otimes \mathbf{L} \mapsto \mathbf{L}$ called Lie brackets which satisfy the so called Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad \forall X, Y, Z \in \mathbf{L}.$$

A Lie subalgebra is a subspace $\mathbf{L}_0 \subset \mathbf{L}$ such that $[\mathbf{L}_0, \mathbf{L}_0] \subset \mathbf{L}_0$. An \mathbb{F} -linear map $U : \mathbf{L}_1 \mapsto \mathbf{L}_2$

between two Lie algebras \mathbf{L}_1 and \mathbf{L}_2 with Lie brackets $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$, respectively, is called a homomorphism if

$$U[X, Y]_1 = [UX, UY]_2, \quad \forall X, Y \in \mathbf{L}_1.$$

If in addition U is bijective then it is called an isomorphism, and the two Lie algebras are called isomorphic, $\mathbf{L}_1 \simeq \mathbf{L}_2$. A Lie subalgebra $\mathbf{I} \subset \mathbf{L}$ is an ideal if $[\mathbf{I}, \mathbf{L}] \subset \mathbf{I}$.

Proposition 1 *For a Lie algebra \mathbf{L} and an ideal $\mathbf{I} \subset \mathbf{L}$, the quotient space \mathbf{L}/\mathbf{I} with Lie brackets induced by the quotient map $q_{\mathbf{I}} : \mathbf{L} \mapsto \mathbf{L}/\mathbf{I}$ is a Lie algebra, and $q_{\mathbf{I}}$ is a Lie algebra homomorphism onto.*

Proof: That \mathbf{I} is an ideal implies

$$[X + \mathbf{I}, Y + \mathbf{I}] \subset [X, Y] + \mathbf{I}, \quad \forall X, Y \in \mathbf{L}.$$

The rest is straightforward. \square

The following is referred to as the First Isomorphism Theorem for Lie algebras.

Theorem 1 *For a Lie algebra homomorphism $U : \mathbf{L}_1 \mapsto \mathbf{L}_2$ between two Lie algebras \mathbf{L}_1 and \mathbf{L}_2 the following assertions hold:*

- *The image $U\mathbf{L}_1 \subset \mathbf{L}_2$ is a Lie subalgebra*
- *The kernel $\ker U \subset \mathbf{L}_1$ is an ideal*
- *The image is isomorphic to the quotient, $U\mathbf{L}_1 \simeq \mathbf{L}_1 / \ker U$*

Proof: For every $X, Y \in \mathbf{L}_1$ we have $[X, Y] \in \mathbf{L}_1$ and $[UX, UY] = U[X, Y] \in U\mathbf{L}_1$ showing that $U\mathbf{L}_1$ is a Lie subalgebra. If in addition $Y \in \ker U$ then $U[X, Y] = [UX, UY] = 0$ implying that $[X, Y] \in \ker U$, i.e., $\ker U$ is an ideal. If we now define a map $T : \mathbf{L}_1 / \ker U \mapsto U\mathbf{L}_1$ by setting $T(X + \ker U) = UX, \forall X \in \mathbf{L}_1$, we can easily verify that T is a Lie algebra isomorphism.

\square

If $\mathbf{I} \subset \mathbf{L}$ is an ideal and if there exists a Lie algebra homomorphism $\tau : \mathbf{L}/\mathbf{I} \mapsto \mathbf{L}$ such that $q \circ \tau = \mathbf{1}$ then $\tau(\mathbf{L}/\mathbf{I}) \subset \mathbf{L}$ is a Lie subalgebra of \mathbf{L} which is isomorphic to \mathbf{L}/\mathbf{I} . In this case we write $\mathbf{L}/\mathbf{I} \subset \mathbf{L}$ and

$$\mathbf{L} = \mathbf{L}/\mathbf{I} \rtimes \mathbf{I}$$

and say that \mathbf{L} is the semidirect product of \mathbf{L}/\mathbf{I} and \mathbf{I} .

A representation ρ of a Lie algebra \mathbf{L} on a vector space \mathbf{W} is a Lie algebra homomorphism $\rho : \mathbf{L} \mapsto \text{End}_{\mathbb{F}}(\mathbf{W})$ where the Lie brackets in the endomorphism ring are given by the ordinary

commutator,

$$[V, U] = VU - UV, \quad \forall V, U \in \text{End}_{\mathbb{F}}(\mathbf{W}).$$

A representation is called faithful if it is injective. For every Lie algebra \mathbf{L} the adjoint representation $\text{ad} : \mathbf{L} \mapsto \text{End}_{\mathbb{F}}(\mathbf{L})$ is given by

$$\text{ad}_X Y = [X, Y], \quad \forall X, Y \in \mathbf{L}.$$

That it is indeed a representation can be shown using the Jacobi identity above. The adjoint representation need not be faithful.

A Lie algebra \mathbf{L} is called decomposable if it decomposes into a direct sum $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$ of two commuting non-trivial Lie subalgebras $\mathbf{L}_1 \neq 0$ and $\mathbf{L}_2 \neq 0$. Otherwise \mathbf{L} is called indecomposable.

A Lie algebra \mathbf{L} is called Abelian if $[X, Y] = 0$ for all $X, Y \in \mathbf{L}$. It is called (N -step) nilpotent if the so called lower central series terminates at $N \in \mathbb{N}$,

$$\mathbf{L}_{(N)} = 0, \quad \mathbf{L}_{(0)} = \mathbf{L}, \quad \mathbf{L}_{(n)} = [\mathbf{L}, \mathbf{L}_{(n-1)}], \quad \forall n \in \mathbb{N}.$$

Abelian Lie algebras are thus 1-step nilpotent. The Lie algebra \mathbf{L} is called (N -step) solvable if the so called derived series terminates at $N \in \mathbb{N}$,

$$\mathbf{L}^{(N)} = 0, \quad \mathbf{L}^{(0)} = \mathbf{L}, \quad \mathbf{L}^{(n)} = [\mathbf{L}^{(n-1)}, \mathbf{L}^{(n-1)}], \quad \forall n \in \mathbb{N}.$$

A nilpotent Lie algebra is necessarily solvable because $\mathbf{L}^{(N)} \subset \mathbf{L}_{(N)} = 0$.

The nilradical $\text{nil}(\mathbf{L}) \subset \mathbf{L}$ of a Lie algebra \mathbf{L} is a maximal nilpotent ideal. If it exists then it is unique, because the sum of two nilpotent ideals is again a nilpotent ideal. It always exists when $\dim_{\mathbb{F}} \mathbf{L} < \infty$.

A Lie algebra isomorphism $\psi : \mathbf{L} \mapsto \mathbf{L}$ is called an automorphism of \mathbf{L} . Automorphisms of a given Lie algebra comprise a group denoted by $\text{Aut}(\mathbf{L})$.

A linear operator $D \in \text{End}_{\mathbb{F}}(\mathbf{L})$ on a Lie algebra \mathbf{L} is called a derivation if

$$D[X, Y] = [DX, Y] + [X, DY], \quad \forall X, Y \in \mathbf{L}.$$

The set $\text{Der}(\mathbf{L}) \subset \text{End}_{\mathbb{F}}(\mathbf{L})$ of all derivations of a given Lie algebra \mathbf{L} comprises a Lie subalgebra

of $\text{End}_{\mathbb{F}}(\mathbf{L})$ under commutation. Indeed, if $D_1, D_2 \in \text{Der}(\mathbf{L})$ then

$$\begin{aligned} [D_1, D_2][X, Y] &= D_1 D_2[X, Y] - D_2 D_1[X, Y] = D_1([D_2 X, Y] + [X, D_2 Y]) - D_2([D_1 X, Y] + [X, D_1 Y]) \\ &= [D_1 D_2 X, Y] + [X, D_1 D_2 Y] - [D_2 D_1 X, Y] - [X, D_2 D_1 Y] = [[D_1, D_2]X, Y] + [X, [D_1, D_2]Y], \end{aligned}$$

so that $[D_1, D_2] \in \text{Der}(\mathbf{L})$. For every $X \in \mathbf{L}$, the operator $\text{ad}_X \in \text{End}_{\mathbb{F}}(\mathbf{L})$ is a derivation. To see this we will use the Jacobi identity,

$$\text{ad}_X[Y, Z] = [X, [Y, Z]] = -[Z, [X, Y]] - [Y, [Z, X]] =$$

$$[[X, Y], Z] + [Y, [X, Z]] = [\text{ad}_X Y, Z] + [Y, \text{ad}_X Z], \quad \forall X, Y, Z \in \mathbf{L}.$$

Thus the image of the adjoint representation $\text{ad}_{\mathbf{L}} \subset \text{Der}(\mathbf{L})$ is a Lie subalgebra called the algebra of inner derivations. Those derivations $D \in \text{Der}(\mathbf{L}) \setminus \text{ad}_{\mathbf{L}}$ are called outer derivations.

A linear operator $T \in \text{End}_{\mathbb{F}}(\mathbf{L})$ on a Lie algebra \mathbf{L} is called Lie orthogonal (see [17]) if

$$[TX, TY] = [X, Y], \quad \forall X, Y \in \mathbf{L}.$$

Obviously the set $\mathbf{O}(\mathbf{L}) \subset \text{End}_{\mathbb{F}}(\mathbf{L})$ of all Lie orthogonal operators on a given Lie algebra \mathbf{L} is a monoid under composition with the identity map being the unit.

For a Lie algebra \mathbf{L} consider the tensor algebra $T(\mathbf{L})$ which is a unital associative graded non-commutative \mathbb{F} -algebra, and consider the ideal $\mathbf{I}_{\mathbf{L}} \subset T(\mathbf{L})$ generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y] \in T(\mathbf{L}), \quad \forall X, Y \in \mathbf{L}.$$

The quotient algebra

$$U(\mathbf{L}) \doteq T(\mathbf{L})/\mathbf{I}_{\mathbf{L}}$$

is called the universal enveloping algebra of \mathbf{L} . The natural inclusion $\mathbf{L} \hookrightarrow T(\mathbf{L})$ gives rise to an injective map $\mathfrak{h} : \mathbf{L} \hookrightarrow U(\mathbf{L})$ such that

$$\mathfrak{h}(X)\mathfrak{h}(Y) - \mathfrak{h}(Y)\mathfrak{h}(X) - \mathfrak{h}([X, Y]) = 0, \quad \forall X, Y \in \mathbf{L}.$$

Denote by

$$\mathcal{Z}(U(\mathbf{L})) = \{x \in U(\mathbf{L}) \mid xy = yx, \quad \forall y \in U(\mathbf{L})\}$$

the centre of the universal enveloping algebra.

Let $D \in \text{Der}(\mathbf{L})$ be a derivation, and let $U(\mathbf{L})$ be the universal enveloping algebra. Using

the map $\mathfrak{h} : \mathbf{L} \mapsto \mathbf{U}(\mathbf{L})$ we can consider \mathbf{L} as a subspace of $\mathbf{U}(\mathbf{L})$. We can extend the action of D to $\mathbf{U}(\mathbf{L})$ by requiring it to be a derivation,

$$D(x * y) = Dx * y + x * Dy, \quad \forall x, y \in \mathbf{U}(\mathbf{L}).$$

In doing so we start from extending D to entire $\mathbf{T}(\mathbf{L})$ as a derivation and using the fact that $D\mathbf{I}_{\mathbf{L}} \subset \mathbf{I}_{\mathbf{L}}$ which follows from

$$D(X \otimes Y - Y \otimes X - [X, Y]) = DX \otimes Y - Y \otimes DX - [DX, Y] + X \otimes DY - DY \otimes X - [X, DY].$$

Thus $\text{Der}(\mathbf{L}) \hookrightarrow \text{Der}(\mathbf{U}(\mathbf{L}))$.

3 Structure of almost Abelian Lie algebras

In this section we will start serving our main purpose by studying the structure of almost Abelian Lie algebras. Abelian Lie algebras are trivial objects characterized by their underlying vector space only, and are therefore often denoted by $\mathbf{L} \doteq \mathbb{F}^{\aleph}$ with $\aleph = \dim_{\mathbb{F}} \mathbf{L}$ (the cardinality of an \mathbb{F} -basis). Arguably the next simplest possible Lie algebras are almost Abelian Lie algebras.

Definition 1 *A non-Abelian Lie algebra \mathbf{L} is called almost Abelian if it contains a codimension 1 Abelian subalgebra.*

This implies that $\mathbf{L} = \mathbb{F} \oplus_{\mathbb{F}} \mathbb{F}^{\aleph}$ as \mathbb{F} -vector spaces where $\aleph = \dim_{\mathbb{F}} \mathbf{L} - 1$ and $[\mathbf{L}, \mathbb{F}^{\aleph}] \neq 0$. The following is a minimal modification of Proposition 3.1 in [4].

Proposition 2 *An almost Abelian Lie algebra \mathbf{L} over \mathbb{F} has a codimension 1 Abelian ideal, and is therefore isomorphic to the semidirect product*

$$\mathbf{L} \simeq \mathbb{F} \rtimes \mathbb{F}^{\aleph}, \quad \aleph = \dim_{\mathbb{F}} \mathbf{L} - 1.$$

Proof: The proof of existence of a codimension 1 Abelian ideal \mathbb{F}^{\aleph} is a word-by-word adaptation of that in the above mentioned reference to the infinite dimensional setting. \mathbf{L} is a semidirect product because the quotient Lie algebra $\mathbf{L}/\mathbb{F}^{\aleph} \simeq \mathbb{F}$ is isomorphic to the subalgebra $\mathbb{F} \oplus 0 \subset \mathbf{L}$. \square

Henceforth we will assume $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ for an element $e_0 \in \mathbf{L}$ and the vector space $\mathbf{V} = \mathbb{F}^{\aleph}$, and will choose an \mathbb{F} -basis $\{e_i\}_{i \in \aleph}$ of \mathbf{V} . By Proposition 6 that will come later the codimension 1 Abelian ideal \mathbf{V} is essentially unique and thus (with one notable exception that we will always

bear in mind) no loss of generality occurs in writing $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$. We will write every $X \in \mathbf{L}$ as $X = (t, v) \in \mathbb{F} \oplus_{\mathbb{F}} \mathbf{V}$. The Lie algebra structure of \mathbf{L} is completely determined by

$$[e_0, v] = \text{ad}_{e_0} v, \quad v \in \mathbf{V}.$$

Here $\text{ad}_{e_0} \in \text{End}_{\mathbb{F}}(\mathbf{V})$ may be any nonzero linear operator on \mathbf{V} .

Remark 1 *Thus an almost Abelian Lie algebra is given by a pair $(\mathbf{V}, \text{ad}_{e_0})$ of a vector space \mathbf{V} and a nonzero linear operator $\text{ad}_{e_0} \in \text{End}_{\mathbb{F}}(\mathbf{V}) \setminus \{0\}$. Every such pair gives rise to an almost Abelian Lie algebra, but different pairs can yield isomorphic Lie algebras.*

Denote by $\mathcal{Z}(\mathbf{L})$ the centre of the Lie algebra \mathbf{L} ,

$$\mathcal{Z}(\mathbf{L}) \doteq \{X \in \mathbf{L} \mid [X, \mathbf{L}] = 0\}.$$

Remark 2 *For an almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ we have*

$$\mathcal{Z}(\mathbf{L}) = \ker \text{ad}_{e_0}, \quad [\mathbf{L}, \mathbf{L}] = \text{ad}_{e_0} \mathbf{V}, \quad [[\mathbf{L}, \mathbf{L}], [\mathbf{L}, \mathbf{L}]] = 0,$$

$$[\mathbf{L}, \dots, [\mathbf{L}, \mathbf{L}] \dots] = \text{ad}_{e_0}^n \mathbf{V}, \quad n \in \mathbb{N}.$$

Thus an almost Abelian \mathbf{L} is always 2-step solvable, and it is nilpotent if and only if the operator ad_{e_0} is. The nilradical is $\text{nil}(\mathbf{L}) = \mathbf{L}$ if \mathbf{L} is nilpotent and $\text{nil}(\mathbf{L}) = \mathbf{V}$ otherwise. In the latter case no ambiguity arises because as we will see later (Proposition 6), in the rare case when there are more than one codimension 1 Abelian ideals \mathbf{V} , \mathbf{L} is necessarily nilpotent. The adjoint representation is given by

$$\text{ad}_X = \begin{pmatrix} 0 & 0 \\ -\text{ad}_{e_0} v & t \text{ad}_{e_0} \end{pmatrix}, \quad \forall X = (t, v) \in \mathbf{L}. \quad (1)$$

We observe immediately that if $\ker \text{ad}_{e_0} = \mathcal{Z}(\mathbf{L}) \neq 0$ then ad is not faithful.

Proposition 3 *Every almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ has a faithful representation in $\text{End}_{\mathbb{F}}(\mathbb{F} \oplus_{\mathbb{F}} \mathbf{V})$ given by*

$$X = (t, v) \mapsto \begin{pmatrix} 0 & 0 \\ v & t \text{ad}_{e_0} \end{pmatrix}, \quad \forall X \in \mathbf{L}.$$

Proof: We first observe that this map is \mathbb{F} -linear and injective, and that

$$\begin{pmatrix} 0 & 0 \\ \mathbf{V} & 0 \end{pmatrix} \simeq \mathbf{V}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{F}\text{ad}_{e_0} \end{pmatrix} \simeq \mathbb{F}e_0$$

as Abelian Lie algebras. Then we check that

$$\text{ad}_{e_0} v = [e_0, v] = \left[\begin{pmatrix} 0 & 0 \\ 0 & \text{ad}_{e_0} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ \text{ad}_{e_0} v & 0 \end{pmatrix}$$

which proves the statement. \square

The next proposition describes all possible Lie subalgebras and ideals of a given almost Abelian Lie algebra.

Proposition 4 *Every Lie subalgebra $\mathbf{L}_0 \subset \mathbf{L}$ of an almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ has one of the following forms:*

- An Abelian Lie subalgebra $\mathbf{L}_0 = \mathbf{W}$ with any subspace $\mathbf{W} \subset \mathbf{V}$
- An Abelian Lie subalgebra $\mathbf{L}_0 = \mathbb{F}e_1 \oplus \mathbf{W}$ with any $e_1 \in e_0 + \mathbf{V}$ and any subspace $\mathbf{W} \subset \ker \text{ad}_{e_0}$
- An almost Abelian Lie subalgebra $\mathbf{L}_0 = \mathbb{F}e_1 \rtimes \mathbf{W}$ with any $e_1 \in e_0 + \mathbf{V}$ and any ad_{e_0} -invariant subspace $\mathbf{W} \subset \mathbf{V}$ with $\mathbf{W} \not\subset \ker \text{ad}_{e_0}$

Every ideal $\mathbf{I} \subset \mathbf{L}$ has one of the following forms:

- An Abelian ideal $\mathbf{I} = \mathbf{W}$ with any ad_{e_0} -invariant subspace $\mathbf{W} \subset \mathbf{V}$
- An Abelian ideal $\mathbf{I} = \mathbb{F}e_1 \oplus \mathbf{W}$ with any $e_1 \in e_0 + \mathbf{V}$ and any subspace $\mathbf{W} \subset \ker \text{ad}_{e_0}$ with $\text{ad}_{e_0} \mathbf{V} \subset \mathbf{W}$
- An almost Abelian ideal $\mathbf{I} = \mathbb{F}e_1 \rtimes \mathbf{W}$ with any $e_1 \in e_0 + \mathbf{V}$ and any subspace $\mathbf{W} \subset \mathbf{V}$ with $\text{ad}_{e_0} \mathbf{V} \subset \mathbf{W} \not\subset \ker \text{ad}_{e_0}$

Proof: Let $\mathbf{L}_0 \subset \mathbf{L}$ be an arbitrary subspace. Then either $\mathbf{L}_0 = \mathbf{W} \subset \mathbf{V}$ or $\mathbf{L}_0 = \mathbb{F}e_1 \oplus_{\mathbb{F}} \mathbf{W}$ for some $e_1 \in e_0 + \mathbf{V}$ and a possibly trivial subspace $\mathbf{W} \subset \mathbf{V}$. In the former case \mathbf{L}_0 is readily an Abelian Lie subalgebra. In the latter case, in order for \mathbf{L}_0 to be a Lie subalgebra we need

$$[\mathbb{F}e_1 \oplus_{\mathbb{F}} \mathbf{W}, \mathbb{F}e_1 \oplus_{\mathbb{F}} \mathbf{W}] = \text{ad}_{e_1} \mathbf{W} = \text{ad}_{e_0} \mathbf{W} \subset \mathbf{W}.$$

If $\mathbf{W} \in \ker \text{ad}_{e_0}$ then $\mathbf{L}_0 = \mathbb{F}e_1 \rtimes \mathbf{W} = \mathbb{F}e_1 \oplus \mathbf{W}$ is Abelian. Otherwise $\mathbf{L}_0 = \mathbb{F}e_1 \rtimes \mathbf{W}$ is almost Abelian.

Now let $\mathbf{I} = \mathbf{L}_0 \subset \mathbf{L}$ be a Lie subalgebra as above. In case $\mathbf{I} = \mathbf{W} \subset \mathbf{V}$ then \mathbf{I} is an ideal when

$$[\mathbf{W}, \mathbf{L}] = \text{ad}_{e_0} \mathbf{W} \subset \mathbf{W}.$$

Otherwise if $\mathbf{I} = \mathbb{F}e_1 \rtimes \mathbf{W}$ then it is an ideal when

$$[\mathbf{I}, \mathbf{L}] = [\mathbb{F}e_1 \rtimes \mathbf{W}, \mathbb{F}e_0 \rtimes \mathbf{V}] = \text{ad}_{e_0} \mathbf{V} \subset \mathbf{W}.$$

This completes the proof. \square

Remark 3 *Note that the second kind of Abelian ideals in the proposition above is possible only if \mathbf{L} is 2-step nilpotent.*

In later sections we will observe that the direct sum $\mathbf{L} \oplus \mathbf{W}$ of an almost Abelian Lie algebra \mathbf{L} with an Abelian Lie algebra \mathbf{W} is not qualitatively different from \mathbf{L} , hence it makes sense to separate out Abelian direct factor subalgebras which are called central extensions.

Lemma 1 *If an almost Abelian Lie algebra is decomposable,*

$$\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2,$$

then \mathbf{L}_1 is almost Abelian and \mathbf{L}_2 is Abelian or vice versa.

Proof: Let P_1 and P_2 be projectors onto the subspaces \mathbf{L}_1 and \mathbf{L}_2 , $P_1 + P_2 = \mathbf{1}$. Let further $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ as usual. Then $P_1 \mathbf{V} \subset \mathbf{L}_1$, $P_2 \mathbf{V} \subset \mathbf{L}_2$ and

$$\mathbf{V} = (P_1 + P_2) \mathbf{V} \subset P_1 \mathbf{V} + P_2 \mathbf{V} = P_1 \mathbf{V} \oplus P_2 \mathbf{V}.$$

It follows that $\text{codim}_{\mathbb{F}}(P_1 \mathbf{V} + P_2 \mathbf{V}) \leq 1$ and therefore, without loss of generality, $P_2 \mathbf{V} = \mathbf{L}_2$ and $\text{codim}_{\mathbb{F}} P_1 \mathbf{V} \leq 1$ in \mathbf{L}_1 . Now $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$ implies that $P_1 : \mathbf{L} \mapsto \mathbf{L}_1$ and $P_2 : \mathbf{L} \mapsto \mathbf{L}_2$ are Lie algebra homomorphisms, thus both $P_2 \mathbf{V} = \mathbf{L}_2$ and $P_1 \mathbf{V}$ are Abelian subalgebras whence the assertion follows. \square

Remark 4 *If an almost Abelian Lie algebra decomposes as $\mathbf{L} = \mathbf{L}_0 \oplus_{\mathbb{F}} \mathbf{W}$ for two subspaces $\mathbf{L}_0, \mathbf{W} \subset \mathbf{L}$ such that $[\mathbf{L}, \mathbf{L}] \subset \mathbf{L}_0$ and $\mathbf{W} \subset \mathcal{Z}(\mathbf{L})$ then $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$. Thus if \mathbf{L} is indecomposable then $\mathcal{Z}(\mathbf{L}) \subset [\mathbf{L}, \mathbf{L}]$.*

Proposition 5 *Every almost Abelian Lie algebra \mathbf{L} can be written as*

$$\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W},$$

where \mathbf{L}_0 is an indecomposable almost Abelian Lie subalgebra and \mathbf{W} is a possibly trivial Abelian Lie subalgebra. If

$$\mathbf{L} = \mathbf{L}'_0 \oplus \mathbf{W}'$$

is another such decomposition then $\mathbf{L}_0 \simeq \mathbf{L}'_0$ and $\mathbf{W} \simeq \mathbf{W}'$.

Proof: Choose as \mathbf{W} any complement of $[\mathbf{L}, \mathbf{L}] \cap \mathcal{Z}(\mathbf{L})$ in $\mathcal{Z}(\mathbf{L})$, i.e.,

$$\mathcal{Z}(\mathbf{L}) = ([\mathbf{L}, \mathbf{L}] \cap \mathcal{Z}(\mathbf{L})) \oplus_{\mathbb{F}} \mathbf{W}. \quad (2)$$

Then by Remark 4 we get $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ for some Lie subalgebra $\mathbf{L}_0 \subset \mathbf{L}$, and by Lemma 1 we have that \mathbf{L}_0 is almost Abelian. If \mathbf{L}_0 is decomposable then $\mathbf{L}_0 = \mathbf{L}_1 \oplus \mathbf{W}_1$ with a non-trivial Abelian subalgebra $\mathbf{W}_1 \neq 0$. It follows that $\mathbf{W} \oplus \mathbf{W}_1 \subset \mathcal{Z}(\mathbf{L})$ but from (2) we find that $(\mathbf{W} \oplus \mathbf{W}_1) \cap [\mathbf{L}, \mathbf{L}] \neq 0$ which is a contradiction.

Suppose that $\mathbf{L} = \mathbf{L}'_0 \oplus \mathbf{W}'$ is another decomposition. From Remark 4 we find that

$$\mathcal{Z}(\mathbf{L}) = \mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W} = (\mathcal{Z}(\mathbf{L}_0) \cap [\mathbf{L}_0, \mathbf{L}_0]) \oplus \mathbf{W} = (\mathcal{Z}(\mathbf{L}) \cap [\mathbf{L}, \mathbf{L}]) \oplus \mathbf{W} = (\mathcal{Z}(\mathbf{L}) \cap [\mathbf{L}, \mathbf{L}]) \oplus \mathbf{W}',$$

hence $\mathbf{W} \simeq \mathbf{W}'$. But then

$$\mathbf{L}_0 \oplus \mathbf{W} = \mathbf{L}'_0 \oplus \mathbf{W}' \simeq \mathbf{L}'_0 \oplus \mathbf{W},$$

whence $\mathbf{L}_0 \simeq \mathbf{L}'_0$ by dividing out \mathbf{W} on both sides. \square

We conclude this section by introducing some of the most prominent almost Abelian Lie algebras as examples.

Example 1 Denote by

$$\mathbf{H}_{\mathbb{F}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & p \\ q & 0 & 0 \end{pmatrix} \mid (p, t, q) \in \mathbb{F}^3 \right\}$$

the Heisenberg algebra which is special in many respects. This corresponds to $\mathbf{H}_{\mathbb{F}} = \mathbb{F}e_0 \ltimes \mathbb{F}^2$ with

$$\text{ad}_{e_0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Example 2 Denote by

$$\mathbf{ax} + \mathbf{b}_{\mathbb{F}} = \left\{ \begin{pmatrix} 0 & 0 \\ b & a \end{pmatrix} \mid (a, b) \in \mathbb{F}^2 \right\}$$

the Lie algebra of generators of affine transformations in \mathbb{F}^2 . This is up to isomorphism the only

non-Abelian 2-dimensional Lie algebra. This corresponds to $\mathbf{ax} + \mathbf{b}_\mathbb{F} = \mathbb{F}e_0 \rtimes \mathbb{F}$ with $\text{ad}_{e_0} = \mathbf{1}$.

4 Homomorphisms and classification

This section is devoted to homomorphisms and identification or classification of almost Abelian Lie algebras. We start by showing that for an almost Abelian Lie algebra which is not a central extension of the Heisenberg algebra (i.e., $\mathbf{L} \not\simeq \mathbf{H}_\mathbb{F} \oplus \mathbf{W}$) the form $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ is unique.

Proposition 6 *If an almost Abelian Lie algebra \mathbf{L} has more than one codimension 1 Abelian ideals then $\mathbf{L} \simeq \mathbf{H}_\mathbb{F} \oplus \mathbf{W}$ with \mathbf{W} Abelian.*

Proof: Let $\mathbf{V} \neq \mathbf{V}'$ be two distinct codimension 1 Abelian ideals so that $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V} = \mathbb{F}e_0 \rtimes \mathbf{V}'$. Denote $\mathbf{V}'' \doteq \mathbf{V} \cap \mathbf{V}'$. Then there is an element $v_1 \in \mathbf{V}$ ($v_1' \in \mathbf{V}'$) such that $v_1 \notin \mathbf{V}'$ ($v_1' \notin \mathbf{V}$) and $\mathbf{V} = \mathbb{F}v_1 \oplus_\mathbb{F} \mathbf{V}''$ ($\mathbf{V}' = \mathbb{F}v_1' \oplus_\mathbb{F} \mathbf{V}''$). That $v_1 \notin \mathbf{V}'$ implies that $v_1 = \lambda e_0' + w_1$ for some $\lambda \in \mathbb{F}^*$ and $w_1 \in \mathbf{V}'$. It follows that

$$[e_0', \mathbf{V}''] = \frac{1}{\lambda}[v_1 - w_1, \mathbf{V}''] = 0,$$

i.e., $\mathbf{V}'' \subset \mathcal{Z}(\mathbf{L})$. Because \mathbf{L} is non-Abelian we know that $v_1 \notin \mathcal{Z}(\mathbf{L})$, thus $\mathbf{V}'' = \mathcal{Z}(\mathbf{L})$. Denoting $v_2 \doteq [e_0, v_1] \neq 0$ we find that

$$[\mathbf{L}, \mathbf{L}] = [e_0, \mathbf{V}] = [e_0, \mathbb{F}v_1 \oplus_\mathbb{F} \mathbf{V}''] = \mathbb{F}v_2.$$

Similarly, if we denote $v_2' \doteq [e_0', v_1'] \neq 0$ then $[\mathbf{L}, \mathbf{L}] = \mathbb{F}v_2'$. It follows that $v_2' = \mu v_2$ for some $\mu \in \mathbb{F}^*$. Now $v_2 \in \mathbf{V}$ and $v_2 = \mu^{-1}v_2' \in \mathbf{V}'$ therefore $v_2 \in \mathbf{V} \cap \mathbf{V}' = \mathbf{V}''$. Let $\mathbf{V}'' = \mathbb{F}v_2 \oplus \mathbf{W}$. Then by Remark 4 we get

$$\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}, \quad \mathbf{L}_0 \doteq \mathbb{F}e_0 \rtimes (\mathbb{F}v_1 \oplus_\mathbb{F} \mathbb{F}v_2).$$

Now it is straightforward to verify that

$$\mathbf{L}_0 \ni pe_0 + qv_1 + tv_2 \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & p \\ q & 0 & 0 \end{pmatrix} \in \mathbf{H}_\mathbb{F}, \quad (p, q, t) \in \mathbb{F}^3$$

is a Lie algebra isomorphism. \square

We continue the study of almost Abelian Lie algebras by considering structure preserving maps between them. Our first subject is a Lie algebra homomorphism $\phi : \mathbf{L} \mapsto \mathbf{L}'$ from an almost Abelian Lie algebra \mathbf{L} to another Lie algebra \mathbf{L}' . The image $\phi(\mathbf{L}) \subset \mathbf{L}'$ is a Lie subalgebra, and the rest of \mathbf{L}' is completely irrelevant from the point of view of the homomorphism ϕ . Therefore in studying ϕ it is no loss of generality to assume that it is onto, i.e., $\phi(\mathbf{L}) = \mathbf{L}'$. Denote by $\text{Hom}(\mathbf{L}, \mathbf{L}')$ the \mathbb{F} -vector space of all Lie algebra homomorphisms $\phi : \mathbf{L} \mapsto \mathbf{L}'$ and by $\text{Hom}(\mathbf{L} \twoheadrightarrow \mathbf{L}') \subset \text{Hom}(\mathbf{L}, \mathbf{L}')$ the subset of surjective homomorphisms.

Remark 5 *The image of a homomorphism $\phi : \mathbf{L} \mapsto \mathbf{L}'$ from an almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ is either Abelian or almost Abelian. Indeed, $\phi\mathbf{V} \subset \phi\mathbf{L}$ is obviously an Abelian ideal with $\text{codim } \phi\mathbf{V} \leq 1$.*

Consider a surjective homomorphism $\phi : \mathbf{L} \mapsto \mathbf{L}'$. According to the Fundamental Theorem on Lie algebra homomorphisms (an explanation can be found in [3]) there exists a unique Lie algebra isomorphism $\psi : \mathbf{L}/\ker \phi \mapsto \mathbf{L}'$ such that $\phi = \psi \circ \mathfrak{q}$ where $\mathfrak{q} : \mathbf{L} \mapsto \mathbf{L}/\ker \phi$ is the canonical quotient homomorphism. Conversely, every isomorphism $\psi : \mathbf{L}/\ker \phi \mapsto \mathbf{L}'$ composed with \mathfrak{q} gives a homomorphism $\psi \circ \mathfrak{q} : \mathbf{L} \mapsto \mathbf{L}'$ with the same kernel $\ker \phi$. Moreover, given a fixed isomorphism $\psi_0 : \mathbf{L}/\ker \phi \mapsto \mathbf{L}'$ every other isomorphism $\psi : \mathbf{L}/\ker \phi \mapsto \mathbf{L}'$ can be written as $\psi = \psi_0 \circ \Psi$ where $\Psi \in \text{Aut}(\mathbf{L}/\ker \phi)$ is an automorphism. And every automorphism Ψ gives rise to a new isomorphism $\psi = \psi_0 \circ \Psi$. We conclude in the following remark.

Remark 6 *There is a bijective correspondence between surjective homomorphisms $\phi : \mathbf{L} \mapsto \mathbf{L}'$ from a fixed Lie algebra \mathbf{L} and pairs (\mathbf{I}, Ψ) where $\mathbf{I} = \ker \phi \subset \mathbf{L}$ is an ideal and $\Psi \in \text{Aut}(\mathbf{L}/\mathbf{I})$.*

Let $\phi : \mathbf{L} \mapsto \mathbf{L}'$ be a Lie algebra homomorphism from an almost Abelian Lie algebra \mathbf{L} into an Abelian or almost Abelian Lie algebra \mathbf{L}' . According to Proposition 5 let $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ with \mathbf{L}_0 an indecomposable almost Abelian and \mathbf{W} a possibly trivial Abelian Lie algebra. In a similar fashion let $\mathbf{L}' = \mathbf{L}'_0 \oplus \mathbf{W}'$ where \mathbf{L}'_0 is either a trivial or an indecomposable almost Abelian and \mathbf{W}' is a possibly trivial Abelian Lie algebra. Then ϕ can be broken down into the block form

$$\begin{pmatrix} \mathbf{L}'_0 \\ \mathbf{W}' \end{pmatrix} = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \begin{pmatrix} \mathbf{L}_0 \\ \mathbf{W} \end{pmatrix} \quad (3)$$

Proposition 7 *The following describes the block form (3) of a surjective homomorphism $\phi \in \text{Hom}(\mathbf{L} \twoheadrightarrow \mathbf{L}')$ from an almost Abelian Lie algebras \mathbf{L} onto an Abelian or almost Abelian Lie algebra \mathbf{L}' :*

- $\phi_{00} \in \text{Hom}(\mathbf{L}_0 \twoheadrightarrow \mathbf{L}'_0)$

- $\phi_{01} \in \text{Hom}(\mathbf{W}, \mathcal{Z}(\mathbf{L}'_0))$
- $\phi_{10} \in \text{Hom}(\mathbf{L}_0, \mathbf{W}')$, i.e., $[\mathbf{L}_0, \mathbf{L}_0] \subset \ker \phi_{10}$
- $\phi_{11} \in \text{Hom}(\mathbf{W}, \mathbf{W}')$ such that $\phi_{10}\mathbf{L}_0 + \phi_{11}\mathbf{W} = \mathbf{W}'$

Proof: By surjectivity of ϕ we have $\mathbf{L}'_0 = \phi_{00}\mathbf{L}_0 + \phi_{01}\mathbf{W}$. Then using $[\mathbf{L}_0, \mathbf{W}] = 0$ and the homomorphism condition for ϕ we find that $\phi_{01}\mathbf{W} \subset \mathcal{Z}(\mathbf{L}'_0)$. But because \mathbf{L}'_0 is indecomposable by Remark 4 we have

$$\mathcal{Z}(\mathbf{L}'_0) \subset [\mathbf{L}'_0, \mathbf{L}'_0] = [\phi_{00}\mathbf{L}_0, \phi_{00}\mathbf{L}_0] \subset \phi_{00}\mathbf{L}_0,$$

therefore $\mathbf{L}'_0 = \phi_{00}\mathbf{L}_0 + \phi_{01}\mathbf{W} \subset \phi_{00}\mathbf{L}_0$ proving the surjectivity of ϕ_{00} . The rest is an easy consequence of Remark 6 and the preceding discussion. \square

Consider now a homomorphism onto itself, i.e., $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$.

Lemma 2 *If a surjective homomorphism $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$ is injective, $\ker \phi = 0$, then so is its component $\phi_{00} \in \text{Hom}(\mathbf{L}_0 \mapsto \mathbf{L}_0)$, i.e., $\ker \phi_{00} = 0$.*

Proof: If $\ker \phi = 0$ then the only solution of the equation

$$\phi X = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_{00}X \\ \phi_{10}X \end{pmatrix} = 0, \quad X \in \mathbf{L}_0$$

is $X = 0$, which means $\ker \phi_{00} \cap \ker \phi_{10} = 0$. Because $\ker \phi_{00} \subset \mathbf{L}_0$ is an ideal this implies

$$[\mathbf{L}_0, \ker \phi_{00}] \subset \ker \phi_{00} \cap [\mathbf{L}_0, \mathbf{L}_0] \subset \ker \phi_{00} \cap \ker \phi_{10} = 0, \quad (4)$$

where we used $[\mathbf{L}_0, \mathbf{L}_0] \subset \ker \phi_{10}$ from Proposition 7. Thus $\ker \phi_{00} \subset \mathcal{Z}(\mathbf{L}_0)$. But \mathbf{L}_0 is indecomposable, hence

$$\ker \phi_{00} \subset \mathcal{Z}(\mathbf{L}_0) \subset [\mathbf{L}_0, \mathbf{L}_0],$$

and thereby using (4) we get

$$\ker \phi_{00} = \ker \phi_{00} \cap [\mathbf{L}_0, \mathbf{L}_0] = 0$$

which completes the proof. \square

Lemma 3 *For a surjective homomorphism $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$, if the component $\phi_{00} \in \text{Hom}(\mathbf{L}_0 \mapsto \mathbf{L}_0)$ is injective, $\ker \phi_{00} = 0$, then $\ker \phi \subset \mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$.*

Proof: Let $X + Y \in \ker \phi \subset \mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$, that is,

$$\phi(X + Y) = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \phi_{00}X + \phi_{01}Y \\ \phi_{10}X + \phi_{11}Y \end{pmatrix} = 0, \quad X \in \mathbf{L}_0, \quad Y \in \mathbf{W}.$$

By Proposition 7 we have $\phi_{00}X = -\phi_{01}Y \in \mathcal{Z}(\mathbf{L}_0)$. This means

$$0 = [\phi_{00}X, \mathbf{L}_0] = [\phi_{00}X, \phi_{00}\mathbf{L}_0] = \phi_{00}[X, \mathbf{L}_0]$$

whence

$$[X, \mathbf{L}_0] \in \ker \phi_{00} = 0,$$

that is, $X \in \mathcal{Z}(\mathbf{L}_0)$. \square

Lemma 4 *Let $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$ with $\ker \phi \subset \mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$. Then $\ker \phi = 0$ if and only if $\ker \phi_{00} \cap \mathcal{Z}(\mathbf{L}_0) = 0$ and $\ker \phi_{11} = 0$.*

Proof: As $\ker \phi \subset \mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$ we have that ϕ is injective if and only if its restriction to $\mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$ is. Because \mathbf{L}_0 is indecomposable and thus by Remark 4

$$\mathcal{Z}(\mathbf{L}_0) \subset [\mathbf{L}_0, \mathbf{L}_0] \subset \ker \phi_{10},$$

the restriction of ϕ to $\mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$ has the form

$$\begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \begin{pmatrix} \mathcal{Z}(\mathbf{L}_0) \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \phi_{00} & \phi_{01} \\ 0 & \phi_{11} \end{pmatrix} \begin{pmatrix} \mathcal{Z}(\mathbf{L}_0) \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \phi_{11} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \phi_{01} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \phi_{00} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathcal{Z}(\mathbf{L}_0) \\ \mathbf{W} \end{pmatrix}.$$

From here it is clear that $\ker \phi = 0$ if and only if $\ker \phi_{00} \cap \mathcal{Z}(\mathbf{L}_0) = 0$ and $\ker \phi_{11} = 0$. \square

We are finally ready to describe the automorphism group $\text{Aut}(\mathbf{L})$ of an almost Abelian Lie algebra \mathbf{L} .

Proposition 8 *The automorphism group of an almost Abelian Lie algebra $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ with \mathbf{L}_0 indecomposable has the following form,*

$$\text{Aut}(\mathbf{L}) = \left\{ \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L}) \left| \begin{array}{l} \phi_{00} \in \text{Aut}(\mathbf{L}_0), \quad \ker \phi_{11} = 0 \end{array} \right. \right\}.$$

Proof: Let $\phi \in \text{Aut}(\mathbf{L})$, i.e., $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$ and $\ker \phi = 0$. Then by Lemma 2 $\ker \phi_{00} = 0$ (i.e., $\phi_{00} \in \text{Aut}(\mathbf{L}_0)$) and by Lemma 4 $\ker \phi_{11} = 0$. Conversely, let $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L})$ such that

$\ker \phi_{00} = \ker \phi_{11} = 0$. Then by Lemma 3 we have $\ker \phi \subset \mathcal{Z}(\mathbf{L}_0) \oplus \mathbf{W}$, and then by Lemma 4 we establish that $\ker \phi = 0$. \square

Homomorphisms between Abelian Lie algebras are simply linear operators. In Proposition 4 we have classified all subalgebras and ideals of almost Abelian Lie algebras. By the above proposition we only need to study $\text{Hom}(\mathbf{L}_0 \mapsto \mathbf{L}'_0)$ in order to have a full understanding of $\text{Hom}(\mathbf{L}, \mathbf{L}')$.

Remark 7 *If $\phi \in \text{Hom}(\mathbf{L} \mapsto \mathbf{L}')$ and $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ then $\mathbf{L}' = \mathbb{F}\phi e_0 \rtimes \phi \mathbf{V}$ and $\phi \circ \text{ad}_{e_0} = \text{ad}_{\phi e_0} \circ \phi$.*

By Remark 6 all we need to do is to study $\text{Aut}(\mathbf{L})$ for indecomposable almost Abelian Lie algebras \mathbf{L} (for Abelian Lie algebras $\text{Aut}(\mathbf{L})$ is the group of invertible operators). For an almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ the action of an endomorphism $\phi \in \text{End}_{\mathbb{F}}(\mathbf{L})$ on an element $X = (t, v) \in \mathbb{F} \oplus_{\mathbb{F}} \mathbf{V}$ can be written in the block form

$$\phi X = \begin{pmatrix} \alpha & \beta^\top \\ \gamma & \Delta \end{pmatrix} \begin{pmatrix} t \\ v \end{pmatrix}, \quad \alpha \in \mathbb{F}, \quad \beta^\top \in \mathbf{V}^*, \quad \gamma \in \mathbf{V}, \quad \Delta \in \text{End}_{\mathbb{F}}(\mathbf{V}). \quad (5)$$

By \mathbf{V}^* (unlike \mathbb{F}^* for a field) we denote the space of linear functionals on \mathbf{V} , i.e., the dual space. The automorphism group of the Heisenberg algebra $\text{Aut}(\mathbf{H}_{\mathbb{F}})$ is a classical subject (e.g., [6], [19]) and we include a proposition merely for completeness.

Proposition 9 *The automorphism group of the Heisenberg algebra in the coordinates (p, t, q) of Example 1 is*

$$\text{Aut}(\mathbf{H}_{\mathbb{F}}) = \left\{ \begin{pmatrix} \alpha & 0 & \beta_2 \\ \gamma_1 & \alpha\Delta_{22} - \beta_2\gamma_2 & \Delta_{12} \\ \gamma_2 & 0 & \Delta_{22} \end{pmatrix} \mid \alpha, \beta_2, \gamma_1, \gamma_2, \Delta_{12}, \Delta_{22} \in \mathbb{F}, \quad \alpha\Delta_{22} - \beta_2\gamma_2 \neq 0 \right\}.$$

Proof: Simply plug in the block form (5) into the homomorphism condition $\phi[X, Y] = [\phi X, \phi Y]$ for all $X, Y \in \mathbf{H}_{\mathbb{F}}$, then require invertibility of ϕ . \square

Automorphism groups of all other indecomposable almost Abelian Lie algebras are more restricted in view of Proposition 6.

Proposition 10 *The automorphism group of an indecomposable almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ other than $\mathbf{H}_{\mathbb{F}}$ is*

$$\text{Aut}(\mathbf{L}) = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \Delta \end{pmatrix} \mid \alpha \in \mathbb{F}^*, \quad \gamma \in \mathbf{V}, \quad \Delta \in \text{Aut}(\mathbf{V}), \quad \Delta \text{ad}_{e_0} - \alpha \text{ad}_{e_0} \Delta = 0 \right\}.$$

Proof: Let $\phi : \mathbf{V} \mapsto \mathbf{V}$ be an automorphism in the block form (5). Then $\phi v = \beta^\top v e_0 + \Delta v$ for $\forall v \in \mathbf{V}$. Because ϕ is bijective we know that $\phi \mathbf{V} \subset \mathbf{L}$ is a codimension 1 Abelian ideal, and as \mathcal{L} is not isomorphic to the Heisenberg algebra we conclude by Proposition 6 that $\phi \mathbf{V} = \mathbf{V}$. Thus $\beta^\top = 0$. Write

$$\phi = \begin{pmatrix} \alpha & 0 \\ \gamma & \Delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}.$$

ϕ is bijective if and only if all three factors are, which is equivalent to $\alpha \neq 0$ and Δ being bijective. Finally the homomorphism condition for ϕ reads

$$\phi \text{ad}_{e_0} v = \Delta \text{ad}_{e_0} v = \phi[e_0, v] = [\phi e_0, \phi v] = [\alpha e_0 + \gamma, \Delta v] = \alpha \text{ad}_{e_0} \Delta v, \quad \forall v \in \mathbf{V},$$

precisely as in the statement. \square

Corollary 1 *For any two codimension 1 Abelian ideals $\mathbf{V}, \mathbf{V}' \subset \mathbf{L}$ of an almost Abelian Lie algebra \mathbf{L} there exists an automorphism $\phi \in \text{Aut}(\mathbf{L})$ such that $\phi \mathbf{V} = \mathbf{V}'$.*

Proof: By Proposition 6 either $\mathbf{V} = \mathbf{V}'$ and we take $\phi = \mathbf{1}$ or $\mathbf{L} = \mathbf{H}_{\mathbb{F}} \oplus \mathbf{W}$ which we will assume. If $\mathbf{L} = \mathbb{F} e_0 \rtimes \mathbf{V}$ then we write $\mathbf{H}_{\mathbb{F}} = \mathbb{F} e_0 \rtimes (\mathbb{F} v_1 \oplus_{\mathbb{F}} \mathbb{F} v_2)$ with $[e_0, v_1] = v_2$ and $[e_0, v_2] = 0$. Thus $\mathbf{V} = \mathbb{F} v_1 \oplus \mathbb{F} v_2 \oplus \mathbf{W}$. By Proposition 4 a codimension 1 Abelian ideal other than \mathbf{V} must have the form $\mathbf{V}' = \mathbb{F} e_1 \oplus \mathbb{F} v_2 \oplus \mathbf{W}$ with $e_1 = e_0 + \lambda v_1$ for some $\lambda \in \mathbb{F}$. In the basis (e_0, v_2, v_1) of Proposition 9 we have

$$\mathbb{F} v_1 \oplus \mathbb{F} v_2 = \left\{ \begin{pmatrix} 0 \\ \mu \\ \nu \end{pmatrix}, \quad (\mu, \nu) \in \mathbb{F}^2 \right\}, \quad \mathbb{F} e_1 \oplus \mathbb{F} v_2 = \left\{ \begin{pmatrix} \eta \\ \rho \\ \lambda \eta \end{pmatrix}, \quad (\eta, \rho) \in \mathbb{F}^2 \right\}.$$

Now it is easy to check that the automorphism $\phi = \psi \oplus \mathbf{1}$ with

$$\psi = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \lambda - 1 & 0 & \lambda \end{pmatrix}$$

will do the job. \square

Our final task will be the classification of almost Abelian Lie algebras into isomorphism classes. Clearly, two Lie algebras $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ and $\mathbf{L}' = \mathbf{L}_0' \oplus \mathbf{W}'$ are isomorphic if and only if $\mathbf{L}_0 \simeq \mathbf{L}_0'$ and $\mathbf{W} \simeq \mathbf{W}'$. It turns out that isomorphism classes of indecomposable almost

Abelian Lie algebras $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ correspond to similarity classes of the operators ad_{e_0} up to rescaling. Recall the description of $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ in terms of the pair $(\mathbf{V}, \text{ad}_{e_0})$ of Remark 1.

Definition 2 *Two pairs (\mathbf{V}_1, T_1) and (\mathbf{V}_2, T_2) with $T_1 \in \text{End}_{\mathbb{F}}(\mathbf{V}_1)$ and $T_2 \in \text{End}_{\mathbb{F}}(\mathbf{V}_2)$ are called similar, $(\mathbf{V}_1, T_1) \sim (\mathbf{V}_2, T_2)$, if there exists an invertible map $\phi : \mathbf{V}_1 \rightarrow \mathbf{V}_2$ such that $\phi T_1 = T_2 \phi$.*

Similarity is an equivalence relation in the class of pairs (\mathbf{V}, T) . For every such pair with $T \neq 0$ we can construct the almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ where $\text{ad}_{e_0} = T$. Conversely, every almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ gives a pair $(\mathbf{V}, \text{ad}_{e_0})$ with $\text{ad}_{e_0} \neq 0$.

Proposition 11 *Two almost Abelian Lie algebras $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ and $\mathbf{L}' = \mathbb{F}e'_0 \rtimes \mathbf{V}'$ are isomorphic if and only if $(\mathbf{V}, \text{ad}_{e_0}) \sim (\mathbf{V}', \lambda \text{ad}_{e'_0})$ for some $\lambda \in \mathbb{F}^*$.*

Proof: Assume first that $(\mathbf{V}, \text{ad}_{e_0}) \sim (\mathbf{V}', \lambda \text{ad}_{e'_0})$, i.e., there is an invertible $\phi : \mathbf{V} \rightarrow \mathbf{V}'$ such that $\phi \text{ad}_{e_0} = \lambda \text{ad}_{e'_0} \phi$. Define the linear invertible map $\Phi : \mathbf{L} \rightarrow \mathbf{L}'$ by setting $\Phi e_0 = \lambda e'_0$ and $\Phi v = \phi v$ for $v \in \mathbf{V}$. It remains to note that Φ is a Lie algebra isomorphism, because

$$\Phi[e_0, v] = \Phi \text{ad}_{e_0} v = \phi \text{ad}_{e_0} v = \lambda \text{ad}_{e'_0} \phi v = [\lambda e'_0, \phi v] = [\Phi e_0, \Phi v], \quad \forall v \in \mathbf{V}.$$

Conversely, let $\Phi : \mathbf{L} \rightarrow \mathbf{L}'$ be a Lie algebra isomorphism. Then $\Phi \mathbf{V} \subset \mathbf{L}'$ is a codimension 1 Abelian ideal, and by Corollary 1 we have an automorphism $\Psi \in \text{Aut}(\mathbf{L}')$ such that $\Psi \Phi \mathbf{V} = \mathbf{V}'$. Now $\Psi \Phi : \mathbf{L} \rightarrow \mathbf{L}'$ is another isomorphism. By Remark 7 we have

$$\mathbf{L}' = \mathbb{F}e'_0 \rtimes \mathbf{V}' = \mathbb{F}\Psi\Phi e_0 \rtimes \Psi\Phi \mathbf{V} = \mathbb{F}\Psi\Phi e_0 \rtimes \mathbf{V}', \quad \Psi\Phi \text{ad}_{e_0} = \text{ad}_{\Psi\Phi e_0} \Psi\Phi.$$

Because $e'_0 \notin \Psi\Phi \mathbf{V} = \mathbf{V}'$ we have that $e'_0 = \lambda \Psi\Phi e_0 + v'_0$ for some $\lambda \in \mathbb{F}^*$ and $v'_0 \in \mathbf{V}'$. It follows that

$$\Psi\Phi \text{ad}_{e_0} = \text{ad}_{\Psi\Phi e_0} \Psi\Phi = \frac{1}{\lambda} \text{ad}_{e'_0} \Psi\Phi,$$

i.e., $(\mathbf{V}, \text{ad}_{e_0}) \sim (\mathbf{V}', \frac{1}{\lambda} \text{ad}_{e'_0})$. \square

The finite dimensional analog of this result for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is pretty straightforward and can be found in the literature (e.g., [7] or [10]). What we have learnt is that isomorphism classes of almost Abelian Lie algebras correspond to the similarity classes of linear operators on vector spaces (\mathbf{V}, T) . A similarity transformation is merely a change of basis. To describe almost Abelian Lie algebras in isomorphism invariant or basis invariant way we need to find a sufficient similarity invariant for linear operators. Different arts of canonical forms can serve as such

invariants. A more explicit description of almost Abelian Lie algebras using canonical forms will be the subject of an upcoming publication.

5 Derivations and Lie orthogonal operators

The present section is devoted to derivations and Lie orthogonal operators of almost Abelian Lie algebras.

Proposition 12 *The algebra of derivations of an almost Abelian Lie algebra $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ with \mathbf{L}_0 indecomposable has the block form*

$$\text{Der}(\mathbf{L}) = \left\{ \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \in \text{End}_{\mathbb{F}}(\mathbf{L}) \mid \phi_{00} \in \text{Der}(\mathbf{L}_0), \quad \phi_{01}\mathbf{W} \subset \mathcal{Z}(\mathbf{L}_0), \quad [\mathbf{L}_0, \mathbf{L}_0] \subset \ker \phi_{10} \right\}.$$

Proof: Take $\forall X, X' \in \mathbf{L}_0$ and $Y, Y' \in \mathbf{W}$, and simply write the derivation condition (the Leibnitz rule)

$$\phi[X + Y, X' + Y'] = \phi[X, X'] = \phi_{00}[X, X'] + \phi_{10}[X, X'] =$$

$$[\phi(X + Y), X' + Y'] + [X + Y, \phi(X' + Y')] = [\phi_{00}X + \phi_{01}Y, X'] + [X, \phi_{00}X' + \phi_{01}Y'].$$

As X, X', Y, Y' are arbitrary we establish the desired result by collecting similar terms. \square

Lemma 5 *Let $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ be an indecomposable almost Abelian Lie algebra. If there exists a nonzero linear functional $\beta^\top \in \mathbf{V}^*$ such that $[\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top$ and*

$$(\beta^\top v)[e_0, v'] = (\beta^\top v')[e_0, v], \quad \forall v, v' \in \mathbf{V}$$

then $\mathbf{L}_0 \simeq \mathbf{H}_{\mathbb{F}}$.

Proof: If $\beta^\top \neq 0$ then $\exists v_0 \in \mathbf{V}$ such that $\beta^\top v_0 \neq 0$. From $[\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top$ we know that $v_0 \notin [\mathbf{L}, \mathbf{L}]$. That \mathbf{L} is indecomposable implies by Remark 4 that $\mathcal{Z}(\mathbf{L}) \subset [\mathbf{L}, \mathbf{L}]$ and therefore $v_0 \notin \mathcal{Z}(\mathbf{L})$, i.e., $[e_0, v_0] \neq 0$. Denoting $v_1 \doteq [e_0, v_0]$ we use the hypothesis of the lemma to write

$$[e_0, \mathbf{V}] = [\mathbf{L}, \mathbf{L}] = \frac{[e_0, v_0]}{\beta^\top v_0} \beta^\top \mathbf{V} = \mathbb{F}v_1$$

and

$$[\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top = \ker \text{ad}_{e_0} = \mathcal{Z}(\mathbf{L}) \subset [\mathbf{L}, \mathbf{L}],$$

whence

$$[\mathbf{L}, \mathbf{L}] = \mathcal{Z}(\mathbf{L}) = \mathbb{F}v_1.$$

But $\ker \beta^\top \subset \mathbf{V}$ is a codimension 1 subspace, so that

$$\mathbf{V} = \mathbb{F}v_0 \oplus_{\mathbb{F}} \ker \beta^\top = \mathbb{F}v_0 \oplus_{\mathbb{F}} \mathcal{Z}(\mathbf{L}) = \mathbb{F}v_0 \oplus_{\mathbb{F}} \mathbb{F}v_1.$$

Now it is clear that $\mathbf{L} = \mathbb{F}e_0 \rtimes (\mathbb{F}v_0 \oplus \mathbb{F}v_1)$ is isomorphic to $\mathbf{H}_{\mathbb{F}}$ (see the final part of the proof of Proposition 6). \square

Proposition 13 *The algebra of derivations of the Heisenberg algebra in the basis of Example 1 is*

$$\text{Der}(\mathbf{H}_{\mathbb{F}}) = \left\{ \begin{pmatrix} \alpha & 0 & \beta_2 \\ \gamma_1 & \alpha + \Delta_{22} & \Delta_{12} \\ \gamma_2 & 0 & \Delta_{22} \end{pmatrix} \mid \alpha, \beta_2, \gamma_1, \gamma_2, \Delta_{12}, \Delta_{22} \in \mathbb{F} \right\}.$$

Proof: Simply plug in the block form (5) into the derivation condition $\phi[X, Y] = [\phi X, Y] + [X, \phi Y]$ for all $X, Y \in \mathbf{H}_{\mathbb{F}}$. \square

Note the similarity between $\text{Aut}(\mathbf{H}_{\mathbb{F}})$ and $\text{Der}(\mathbf{H}_{\mathbb{F}})$: in the component Δ_{11} the determinant of the corresponding algebraic minor is replaced by its trace. In general, when $\text{Aut}(\mathbf{H}_{\mathbb{F}})$ is a Lie group then $\text{Der}(\mathbf{H}_{\mathbb{F}})$ is related to its Lie algebra.

Proposition 14 *The algebra of derivations of an indecomposable almost Abelian Lie algebra $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ is*

$$\text{Der}(\mathbf{L}) = \left\{ \begin{pmatrix} \alpha & \beta^\top \\ \gamma & \Delta \end{pmatrix} \mid \begin{array}{l} \alpha \in \mathbb{F}, \quad \beta^\top \in \mathbf{V}^*, \quad \gamma \in \mathbf{V}, \quad \Delta \in \text{End}_{\mathbb{F}}(\mathbf{V}), \\ [\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top, \quad (\Delta - \alpha \mathbf{1})\text{ad}_{e_0} - \text{ad}_{e_0} \Delta = 0 \end{array} \right\}.$$

If $\mathbf{L} \not\cong \mathbf{H}_{\mathbb{F}}$ then $\beta^\top = 0$.

Proof: We simply substitute the block form (5) into the derivation condition $\phi[X, Y] = [\phi X, Y] + [X, \phi Y]$ for all $X, Y \in \mathbf{H}_{\mathbb{F}}$. Collecting similar terms we get the following three equations,

$$\beta^\top \text{ad}_{e_0} = 0,$$

$$(\beta^\top v) \text{ad}_{e_0} v' - (\beta^\top v') \text{ad}_{e_0} v = 0,$$

$$(\Delta - \alpha \mathbf{1}) \text{ad}_{e_0} - \text{ad}_{e_0} \Delta = 0.$$

The first equation simply means that $\text{ad}_{e_0} \mathbf{V} = [\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top$. It only remains to use Lemma 5 to conclude that if $\mathbf{L} \not\cong \mathbf{H}_{\mathbb{F}}$ then $\beta^\top = 0$. \square

Corollary 2 *If all derivations of an almost Abelian Lie algebra are inner, i.e., $\text{Der}(\mathbf{L}) = \text{ad}_{\mathbf{L}}$, then \mathbf{L} is isomorphic to $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$.*

Proof: If $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ with \mathbf{L}_0 indecomposable as usual, then

$$\text{ad}_{\mathbf{L}} = \text{ad}_{\mathbf{L}_0} \oplus \text{ad}_{\mathbf{W}} = \text{ad}_{\mathbf{L}_0} \oplus 0,$$

that is, the operator ad_{X+Y} in (1) is the block sum of ad_X and zero for every $X \in \mathbf{L}_0$, $Y \in \mathbf{W}$. But from Proposition 12 we know that if $\mathbf{W} \neq 0$ then there always exist non-trivial derivations with $\phi_{00} = \phi_{01} = \phi_{10} = 0$ and $\phi_{11} \neq 0$. Therefore if $\text{Der}(\mathbf{L}) = \text{ad}_{\mathbf{L}}$ then \mathbf{L} is necessarily indecomposable. Then by Proposition 14 we have

$$\begin{pmatrix} 0 & 0 \\ \text{ad}_{e_0} \mathbf{V} & \mathbb{F} \text{ad}_{e_0} \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & \beta^\top \\ \gamma & \Delta \end{pmatrix} \middle| \begin{array}{l} \alpha \in \mathbb{F}, \quad \beta^\top \in \mathbf{V}^*, \quad \gamma \in \mathbf{V}, \quad \Delta \in \text{End}_{\mathbb{F}}(\mathbf{V}), \\ [\mathbf{L}, \mathbf{L}] \subset \ker \beta^\top, \quad (\Delta - \alpha \mathbf{1}) \text{ad}_{e_0} - \text{ad}_{e_0} \Delta = 0 \end{array} \right\}.$$

In particular, for $\alpha = 0$ and $\beta^\top = 0$ the set $\Delta \in \mathbb{F} \text{ad}_{e_0}$ is the full set of solutions of the equation $[\Delta, \text{ad}_{e_0}] = 0$. It is clear that $\mathbb{F} \mathbf{1}$ commutes with everything and hence $\mathbb{F} \mathbf{1} \subset \mathbb{F} \text{ad}_{e_0}$ which means $\text{ad}_{e_0} \in \mathbb{F}^* \mathbf{1}$. But then every $\Delta \in \text{End}_{\mathbb{F}}(\mathbf{V})$ commutes with ad_{e_0} , hence $\mathbb{F} \mathbf{1} = \text{End}_{\mathbb{F}}(\mathbf{V})$. This is possible only when $\dim_{\mathbb{F}} \mathbf{V} = 1$, so that $\mathbf{V} = \mathbb{F}v$ for some $v \neq 0$ such that $[e_0, v] = \lambda v$ with $\lambda \in \mathbb{F}^*$. It is now not difficult to check directly that the map

$$(ae_0, bv) \mapsto \begin{pmatrix} 0 & 0 \\ b & \lambda a \end{pmatrix}, \quad \forall a, b \in \mathbb{F}$$

provides a Lie algebra isomorphism between \mathbf{L} and $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$. \square

Let us now switch to Lie orthogonal operators. The finite dimensional analogs of what follows can be found in [17], but there the authors use matrix terminology which is not directly applicable in our generality. The underlying reasoning, of course, is similar. Recall that $\text{SL}(2, \mathbb{F})$ is the group of 2×2 \mathbb{F} -valued matrices of determinant 1.

Proposition 15 *The Lie orthogonal operators of the Lie algebras $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$ and $\mathbf{H}_{\mathbb{F}}$ are*

$$\mathbf{O}(\mathbf{ax} + \mathbf{b}_{\mathbb{F}}) = \text{SL}(2, \mathbb{F}),$$

$$\mathbf{O}(\mathbf{H}_{\mathbb{F}}) = \left\{ \begin{pmatrix} \alpha & 0 & \beta_2 \\ \gamma_1 & \Delta_{11} & \Delta_{12} \\ \gamma_2 & 0 & \Delta_{22} \end{pmatrix} \middle| \alpha, \beta_2, \gamma_1, \gamma_2, \Delta_{11}, \Delta_{12}, \Delta_{22} \in \mathbb{F}, \quad \alpha \Delta_{22} - \gamma_2 \beta_2 = 1 \right\}.$$

Proof: These results can be obtain straightforwardly by a direct substitution of the matrix form (5) into the Lie orthogonality condition $[\phi X, \phi Y] = [X, Y]$ for all X and Y . \square

Lemma 6 *For an indecomposable almost Abelian Lie algebra \mathbf{L} if $\dim_{\mathbb{F}}[\mathbf{L}, \mathbf{L}] = 1$ then either $\mathbf{L} \simeq \mathbf{ax} + \mathbf{b}_{\mathbb{F}}$ or $\mathbf{L} \simeq \mathbf{H}_{\mathbb{F}}$.*

Proof: Let $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ and let $v_0 \in \mathbf{V}$ such that $[\mathbf{L}, \mathbf{L}] = \mathbb{F}v_0$. There exists a nonzero $v_1 \in \mathbf{V}$ such that $[e_0, v_1] = v_0$. For an arbitrary $v \in \mathbf{V}$, $[e_0, v] = \mu(v)v_0$ for a linear functional $\mu \in \mathbf{V}^*$. It follows that $[e_0, v - \mu(v)v_1] = 0$ which means that $v - \mu(v)v_1 \in \mathcal{Z}(\mathbf{L})$ for all $v \in \mathbf{V}$. This in turn implies

$$\mathbf{V} = \mathbb{F}v_1 \oplus_{\mathbb{F}} \mathcal{Z}(\mathbf{L}). \quad (6)$$

Because \mathbf{L} is indecomposable we have $\mathcal{Z}(\mathbf{L}) \subset \mathbb{F}v_0$. We have two possibilities. If $\mathcal{Z}(\mathbf{L}) = 0$ then $v_0 \notin \mathcal{Z}(\mathbf{L})$ and therefore $[e_0, v_0] = \lambda v_0$ for some $\lambda \in \mathbb{F}^*$, i.e., $v_1 = \frac{1}{\lambda}v_0$. By (6) we get $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbb{F}v_0$ which is isomorphic to $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$ (compare with the final part of the proof of Corollary 2). Otherwise, if $\mathcal{Z}(\mathbf{L}) = \mathbb{F}v_0$ then again by (6) we have $\mathbf{L} = \mathbb{F}e_0 \rtimes (\mathbb{F}v_0 \oplus_{\mathbb{F}} \mathbb{F}v_1)$ which is isomorphic to $\mathbf{H}_{\mathbb{F}}$ (compare with the final part of the proof of Proposition 6). \square

Proposition 16 *If an indecomposable almost Abelian Lie algebra \mathbf{L} is not isomorphic to either $\mathbf{ax} + \mathbf{b}_{\mathbb{F}}$ or $\mathbf{H}_{\mathbb{F}}$ then its Lie orthogonal operators are*

$$\mathbf{O}(\mathbf{L}) = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \frac{1}{\alpha}\mathbf{1} + Z \end{pmatrix} \mid \alpha \in \mathbb{F}^*, \quad \gamma \in \mathbf{V}, \quad Z \in \text{Hom}(\mathbf{V}, \ker \text{ad}_{e_0}) \right\}.$$

Proof: As usual, we start by substituting the matrix form (5) into the Lie orthogonality condition,

$$\left[\begin{pmatrix} \alpha & \beta^{\top} \\ \gamma & \Delta \end{pmatrix} \begin{pmatrix} t \\ v \end{pmatrix}, \begin{pmatrix} \alpha & \beta^{\top} \\ \gamma & \Delta \end{pmatrix} \begin{pmatrix} t' \\ v' \end{pmatrix} \right] = \left[\begin{pmatrix} t \\ v \end{pmatrix}, \begin{pmatrix} t' \\ v' \end{pmatrix} \right] = \begin{pmatrix} 0 \\ t \text{ad}_{e_0} v' - t' \text{ad}_{e_0} v. \end{pmatrix}.$$

This is equivalent to the following system of two equations,

$$(\beta^{\top} v) \text{ad}_{e_0} \Delta v' = (\beta^{\top} v') \text{ad}_{e_0} \Delta v, \quad (7)$$

$$\alpha \text{ad}_{e_0} \Delta v = (\beta^{\top} v) \text{ad}_{e_0} \gamma + \text{ad}_{e_0} v, \quad \forall v, v' \in \mathbf{V}. \quad (8)$$

Suppose first that $\beta^{\top} \neq 0$, which means that there exists $v_0 \in \mathbf{V}$ such that $\beta^{\top} v_0 \neq 0$. From (7) we get

$$\text{ad}_{e_0} \Delta = \frac{\text{ad}_{e_0} \Delta v_0}{\beta^{\top} v_0} \beta^{\top},$$

and combining with (8) we find that

$$\text{ad}_{e_0} = \left(\frac{\alpha \text{ad}_{e_0} \Delta v_0}{\beta^\top v_0} - \text{ad}_{e_0} \gamma \right) \beta^\top.$$

It follows that $[\mathbf{L}, \mathbf{L}] = \text{ad}_{e_0} \mathbf{V}$ is 1-dimensional (\mathbf{L} is non-Abelian hence $\text{ad}_{e_0} \neq 0$). According to Lemma 6 we conclude that \mathbf{L} is either $\mathbf{ax} + \mathbf{b}_\mathbb{F}$ or $\mathbf{H}_\mathbb{F}$ which contradicts the synopsis of this proposition. Therefore we necessarily have $\beta^\top = 0$. Now (7) becomes trivial and (8) simplifies to

$$\text{ad}_{e_0} = \alpha \text{ad}_{e_0} \Delta.$$

This is equivalent to saying that $\alpha \neq 0$ and $\text{ad}_{e_0}(\Delta - \frac{1}{\alpha} \mathbf{1}) = 0$. Denoting $Z \doteq \Delta - \frac{1}{\alpha} \mathbf{1}$ we see that $Z\mathbf{V} \subset \ker \text{ad}_{e_0} = \mathcal{Z}(\mathbf{L})$ which completes the proof. \square

Corollary 3 *If \mathbf{L} is an indecomposable almost Abelian Lie algebra and if $\phi \in \mathbf{O}(\mathbf{L})$ is any Lie orthogonal operator then $\phi\mathbf{L} + \mathcal{Z}(\mathbf{L}) = \mathbf{L}$.*

Proof: By Proposition 15 the claim is obvious for $\mathbf{ax} + \mathbf{b}_\mathbb{F}$ as all Lie orthogonal operators are bijective. For $\mathbf{H}_\mathbb{F}$ a Lie orthogonal operator ϕ in the form presented in Proposition 15 is non-surjective when $\Delta_{11} = 0$, but it is precisely the centre $\mathcal{Z}(\mathbf{H}_\mathbb{F})$ that misses in the image $\phi\mathbf{H}_\mathbb{F}$ and therefore again $\mathbf{H}_\mathbb{F} = \phi\mathbf{H}_\mathbb{F} + \mathcal{Z}(\mathbf{H}_\mathbb{F})$. Now assume \mathbf{L} is not one of the two algebras above. Then by Proposition 16 we have that

$$\phi\mathbf{L} = \mathbb{F}e_0 \oplus_\mathbb{F} \left(\mathbb{F}\gamma + \left(\frac{1}{\alpha} \mathbf{1} + Z \right) \mathbf{V} \right).$$

The identity

$$v = \left(\frac{1}{\alpha} \mathbf{1} + Z \right) \alpha v - \alpha Zv, \quad \forall v \in \mathbf{V}$$

along with the fact that $Z\mathbf{V} \subset \mathcal{Z}(\mathbf{L})$ show that

$$\mathbf{V} = \left(\frac{1}{\alpha} \mathbf{1} + Z \right) \mathbf{V} + \mathcal{Z}(\mathbf{L})$$

which completes the proof. \square

Proposition 17 *For an almost Abelian Lie algebra $\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{W}$ with \mathbf{L}_0 indecomposable the Lie orthogonal operators have the following block form,*

$$\mathbf{O}(\mathbf{L}) = \left\{ \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix} \in \text{End}_\mathbb{F}(\mathbf{L}) \left| \begin{array}{l} \phi_{00} \in \mathbf{O}(\mathbf{L}_0), \quad \phi_{01} \mathbf{W} \subset \mathcal{Z}(\mathbf{L}_0) \end{array} \right. \right\}.$$

Proof: We write down the Lie orthogonality condition $[\phi(X+Y), \phi(X'+Y')] = [X+Y, X'+Y']$ for arbitrary $X, X' \in \mathbf{L}_0$ and $Y, Y' \in \mathbf{W}$. Collecting similar terms we get the following pair of equations,

$$[\phi_{00}X, \phi_{00}X'] = [X, X'], \quad [\phi_{01}\mathbf{W}, \phi_{00}\mathbf{L}_0 + \phi_{01}\mathbf{W}] = 0.$$

The first equation simply means $\phi_{00} \in \mathbf{O}(\mathbf{L}_0)$. The second equation in view of Corollary 3 gives

$$[\phi_{01}\mathbf{W}, \mathbf{L}_0] = [\phi_{01}\mathbf{W}, \phi_{00}\mathbf{L}_0 + \mathcal{Z}(\mathbf{L}_0)] = [\phi_{01}\mathbf{W}, \phi_{00}\mathbf{L}_0] = 0$$

whence we get $\phi_{01}\mathbf{W} \subset \mathcal{Z}(\mathbf{L}_0)$. \square

6 The centre of the universal enveloping algebra

Here we are going to find the centre of the universal enveloping algebra of an almost Abelian Lie algebra \mathbf{L} . If $\mathfrak{h} : \mathbf{L} \mapsto \mathbf{U}(\mathbf{L})$ is the embedding map and $\mathbf{L} = \mathbb{F}e_0 \rtimes \mathbf{V}$ is an almost Abelian Lie algebra, then let us choose a basis $\{e_i\}_{i \in \mathbb{N}}$ of \mathbf{V} and denote

$$x_0 = \mathfrak{h}(e_0), \quad x_i = \mathfrak{h}(e_i), \quad \forall i \in \mathbb{N}.$$

Then $x_i x_j = x_j x_i$ for all $i, j \in \mathbb{N}$. By Poincaré-Birkhoff-Witt theorem the monomials of the form $x_0^n P(x_{\mathbb{N}})$ with $P(x_{\mathbb{N}}) \in \mathbb{F}[\{x_i\}_{i \in \mathbb{N}}]$ span $\mathbf{U}(\mathbf{L})$. In other words, every element $x \in \mathbf{U}(\mathbf{L})$ has a unique representation

$$x = \sum_{m=0}^n x_0^m P_m(x_{\mathbb{N}}), \quad n \in \mathbb{N}_0, \quad P_m(x_{\mathbb{N}}) \in \mathbb{F}[\{x_i\}_{i \in \mathbb{N}}].$$

Denote by $\text{ad} \in \text{Der}(\mathbf{U}(\mathbf{L}))$ the extension of the derivation $\text{ad}_{e_0} \in \text{Der}(\mathbf{L})$. For ad the variable x_0 is a constant,

$$\text{ad}x = \sum_{m=0}^n x_0^m \text{ad}P_m(x_{\mathbb{N}}),$$

and on every $P(x_{\mathbb{N}})$ is acts as a derivation,

$$\text{ad}(x_{i_1} x_{i_2} \dots x_{i_p}) = (\text{ad}x_{i_1})x_{i_2} \dots x_{i_p} + x_{i_1}(\text{ad}x_{i_2}) \dots x_{i_p} + \dots + x_{i_1} x_{i_2} \dots (\text{ad}x_{i_p}).$$

In particular, ad preserves the degree of every homogeneous polynomial (zero monomial is of any degree of homogeneity).

Lemma 7 For every $n \in \mathbb{N}_0$ and $P(x_{\mathbb{N}}) \in \mathbb{F}[\{x_i\}_{i \in \mathbb{N}}]$ we have

$$P(x_{\mathbb{N}})x_0^n = (x_0 - \text{ad})^n P(x_{\mathbb{N}}).$$

Proof: We start by noting that

$$x_i x_0 = x_0 x_i - \text{ad} x_i, \quad \forall i \in \mathbb{N}.$$

Assume that for every monomial of degree q less than some $p \in \mathbb{N}$ it holds

$$x_{i_1} \dots x_{i_q} x_0 = (x_0 - \text{ad})(x_{i_1} \dots x_{i_q}).$$

Then for every monomial of degree p we have

$$\begin{aligned} x_{i_1} \dots x_{i_p} x_0 &= x_{i_1} (x_0 - \text{ad})(x_{i_2} \dots x_{i_p}) = x_{i_1} x_0 x_{i_2} \dots x_{i_p} - x_{i_1} \text{ad}(x_{i_2} \dots x_{i_p}) = \\ &= (x_0 - \text{ad})(x_{i_1}) x_{i_2} \dots x_{i_p} - x_{i_1} \text{ad}(x_{i_2} \dots x_{i_p}) = (x_0 - \text{ad}) x_{i_1} \dots x_{i_p}. \end{aligned}$$

Thus by mathematical induction we prove the statement for all $p \in \mathbb{N}$. Every polynomial is a sum of monomials, hence

$$P(x_{\mathbb{N}})x_0 = (x_0 - \text{ad})P(x_{\mathbb{N}}), \quad \forall P(x_{\mathbb{N}}) \in \mathbb{F}[\{x_i\}_{i \in \mathbb{N}}].$$

Now assume that

$$P(x_{\mathbb{N}})x_0^m = (x_0 - \text{ad})^m P(x_{\mathbb{N}})$$

holds for all $P(x_{\mathbb{N}})$ and $m < n$ for a fixed $n \in \mathbb{N}$. Then

$$P(x_{\mathbb{N}})x_0^n = P(x_{\mathbb{N}})x_0^{n-1}x_0 = (x_0 - \text{ad})^{n-1}P(x_{\mathbb{N}})x_0 = (x_0 - \text{ad})^n P(x_{\mathbb{N}}).$$

Using mathematical induction once more we arrive at the desired result. \square

Proposition 18 The centre of the universal enveloping algebra of an almost Abelian Lie algebra \mathbf{L} as above consists of ad -conserved polynomials in $\{x_i\}_{i \in \mathbb{N}}$,

$$\mathcal{Z}(\mathbf{U}(\mathbf{L})) = \{P(x_{\mathbb{N}}) \in \mathbb{F}[\{x_i\}_{i \in \mathbb{N}}] \mid \text{ad}P(x_{\mathbb{N}}) = 0\}.$$

Proof: Let $x \in \mathbf{U}(\mathbf{L})$ be an arbitrary element of the universal enveloping algebra,

$$x = x_0^n P_n(x_{\aleph}) + \dots + P_0(x_{\aleph}), \quad n \in \mathbf{N}_0, \quad P_m(x_{\aleph}) \in \mathbb{F}[\{x_i\}_{i \in \aleph}].$$

The condition that $x \in \mathcal{Z}(\mathbf{U}(\mathbf{L}))$ is equivalent to the following set of two conditions,

$$xx_0 = x_0x, \quad xx_i = x_ix, \quad \forall i \in \aleph.$$

Using Lemma 7 the first condition can be transformed as

$$xx_0 = (x_0^n P_n(x_{\aleph}) + \dots + P_0(x_{\aleph}))x_0 = x_0^n (x_0 - \text{ad})P_n(x_{\aleph}) + \dots + (x_0 - \text{ad})P_0(x_{\aleph}) =$$

$$x_0^{n+1}P_n(x_{\aleph}) + \dots + x_0P_0(x_{\aleph}) - x_0^n \text{ad}P_n(x_{\aleph}) - \dots - \text{ad}P_0(x_{\aleph}) = x_0x - x_0^n \text{ad}P_n(x_{\aleph}) - \dots - \text{ad}P_0(x_{\aleph}) = x_0x,$$

which is equivalent to

$$\text{ad}P_m(x_{\aleph}) = 0, \quad m = 0, \dots, n.$$

The second condition becomes (again with the help of Lemma 7)

$$x_ix = x_i(x_0^n P_n(x_{\aleph}) + \dots + P_0(x_{\aleph})) = [(x_0 - \text{ad})^n x_i]P_n(x_{\aleph}) + \dots + x_iP_0(x_{\aleph}) =$$

$$\left[x_0^n x_i + \sum_{l=1}^n x_0^{n-l} (-\text{ad})^l x_i \right] P_n(x_{\aleph}) + \dots + x_i P_0(x_{\aleph}) =$$

$$x_0^n P_n(x_{\aleph})x_i + \dots + P_0(x_{\aleph})x_i + \sum_{l=1}^n x_0^{n-l} [(-\text{ad})^l x_i] P_n(x_{\aleph}) + \dots + \sum_{l=1}^1 x_0^{1-l} [(-\text{ad})^l x_i] P_1(x_{\aleph}) =$$

$$x_ix + \sum_{l=1}^n x_0^{n-l} [(-\text{ad})^l x_i] P_n(x_{\aleph}) + \dots + \sum_{l=1}^1 x_0^{1-l} [(-\text{ad})^l x_i] P_1(x_{\aleph}) = x_ix, \quad \forall i \in \aleph,$$

which means

$$\sum_{l=1}^n x_0^{n-l} [(-\text{ad})^l x_i] P_n(x_{\aleph}) + \dots + \sum_{l=1}^1 x_0^{1-l} [(-\text{ad})^l x_i] P_1(x_{\aleph}) =$$

$$x_0^{n-1} [(-\text{ad})x_i] P_n(x_{\aleph}) + x_0^{n-2} ([(-\text{ad})^2 x_i] P_n(x_{\aleph}) + [(-\text{ad})x_i] P_{n-1}(x_{\aleph})) + \dots$$

$$+ [(-\text{ad})^n x_i] P_n(x_i) + \dots + [(-\text{ad})x_i] P_1(x_{\aleph}) = 0, \quad \forall i \in \aleph.$$

Start with the term proportional to x_0^{n-1} . Because \mathbf{L} is non-Abelian, there exists $i \in \aleph$ such that $\text{ad}x_i \neq 0$ and thus $P_n(x_{\aleph}) = 0$. Take next the term proportional to x_0^{n-2} and set $P_n(x_{\aleph}) = 0$:

we get $P_{n-1}(x_{\mathbb{N}}) = 0$. Continuing in a similar fashion we will find that

$$P_m(x_{\mathbb{N}}) = 0, \quad m = 1, \dots, n.$$

Therefore

$$x = P_0(x_{\mathbb{N}}), \quad \text{ad} P_0(x_{\mathbb{N}}) = 0$$

as asserted. \square

Remark 8 *Note that if $X \in \mathcal{Z}(\mathbf{L}) = \ker \text{ad}_{e_0}$ and therefore $x = \mathfrak{h}(X) \in \ker \text{ad}$ then $\mathbb{F}[x] \in \mathcal{Z}(\mathbf{U}(\mathbf{L}))$. We can write this symbolically as $\mathbb{F}[\mathcal{Z}(\mathbf{L})] \subset \mathcal{Z}(\mathbf{U}(\mathbf{L}))$. The next example shows that this inclusion is often proper.*

Example 3 *Let $\text{Bi}(\text{VII}_0) = \mathbb{F}e_0 \rtimes \mathbb{F}^2$ be the Bianchi VII₀ Lie algebra,*

$$\text{Bi}(\text{VII}_0) = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ v & 0 & -t \\ u & t & 0 \end{array} \right) \mid (t, v, u) \in \mathbb{F}^3 \right\}.$$

This corresponds to

$$\text{ad}_{e_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we let $\mathbf{U}(\text{Bi}(\text{VII}_0)) = \mathbb{F}[x_0, x_1, x_2]$ with

$$x_0 = \mathfrak{h} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_1 = \mathfrak{h} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_2 = \mathfrak{h} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

then we have $\text{ad} x_1 = x_2$ and $\text{ad} x_2 = -x_1$. It follows that $\text{ad}(x_1^2 + x_2^2) = 0$, i.e., $x_1^2 + x_2^2 \in \mathcal{Z}(\mathbf{U}(\text{Bi}(\text{VII}_0)))$ although $\mathcal{Z}(\text{Bi}(\text{VII}_0)) = \ker \text{ad}_{e_0} = 0$.

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